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Conformally Flat Space-Times of Locally Constant Connection. I.[1]

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Conformally flat space-times of locally constant connection are studied. The constant connection defines a global vector field which is assumed timelike. The general solution of the geodesic equations is presented and several theorems characterizing the geometry of such space-times are proved.

1. INTRODUCTION

This is the first of two papers dealing with conformally flat space-times in which the conformal factor is a homogeneous linear function of the space-time coordinates. Such manifolds, first studied by Antonelli and Voorhees (1975), have been called spaces of locally constant connection (Ruhnau, 1977). Their particular merit is that it is always possible to find local coordinates in which the connection coefficients are constant. This has the effect of "decoupling" the geodesic equations into the independent pair

$$\frac{dx^i}{dt} = u^i$$

$$\frac{du^i}{dt} = -\Gamma^i_{jk} u^j u^k \tag{1}$$

In particular, it becomes possible to find the general solution of these equations and based on this to obtain some rather elegant results characterizing the geometry of these spaces (Voorhees, 1983).

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Spaces of locally constant connection were first used in the case of positive definite metric to obtain a geometric reformulation of the classical Lotka–Volterra equation of mathematical ecology (Antonelli and Voorhees, 1975) and to model the growth of idealized coral reefs (Antonelli, 1983; Voorhees, 1983). They have been studied by Ruhnau (1977), who proved the following:

> *Theorem 1.* Sufficient conditions for a locally constant Levi-Cività connection Γ^i_{jk} to be a metric connection are that there exist a constant vector γ_k such that in a coordinate system in which the Γ^i_{jk} are constants,
>
> $$\eta_{s(i}\Gamma^s_{l)k} = \gamma_k \eta_{il} \tag{2}$$
>
> where η_{ij} is the appropriate flat metric (i.e., η_{ij} in our case is the Minkowski metric).

She conjectures that (2) is also necessary. If this is correct then the only possible metric for a space of locally constant connection is

$$g_{ij} = e^{2\phi}\eta_{ij}$$

$$\phi = \sum_{k=1}^{n} \alpha^k x^k \tag{3}$$

Only metrics of this form, with $\phi = \alpha^1 x^1 + \alpha^2 x^2 + \alpha^3 x^3 - \alpha^4 x^4 \equiv (\alpha \cdot x)$ will be considered. Ruhnau has also geometrically characterized spaces of locally constant connection in terms of groups of affine transformations, proving the following:

> *Theorem 2* (Ruhnau, 1977). An n-dimensional manifold with metric (M, g) is a space of locally constant connection if and only if it admits an n-parameter transitive Abelian group of affine transformations.

The connection coefficients, Riemann, Ricci, and Einstein tensors, and scalar curvature for a metric (3) are given in the Appendix. The notation is that for any vectors u, v $|u|^2$ and $(u \cdot v)$ indicate the product formed with respect to the Minkowski metric $\eta_{ij} = \mathrm{diag}(1, 1, 1, -1)$, while $u_i u^i$ and $u_i v^i$ indicate the product with respect to the conformal metric g_{ij}. Units are $G = c = 1$.

Since $R_{ij} \neq 0$ the space-time is not empty. By inspection of the Einstein tensor (A.5) we obtain the following:

Theorem 3. Let (M, g) be a space-time with locally constant connection. The stress-energy tensor of (M, g) satisfies the nonnegative energy condition $T_{ij}t^i t^j \geqslant 0$ for all timelike t^i if and only if α^i is timelike or null.

In this paper α^i will be assumed timelike and future pointing. The second paper (in this issue: Conformally Flat Spaces of Locally Constant Connection, II) will consider the case in which α^i is null.

2. GEODESICS

It is not difficult to see that the geodesic equations for a space of locally constant connection can be written as

$$\frac{dx^i}{dt} = u^i$$

$$\frac{du^i}{dt} = |u|^2\alpha^i - 2(\alpha \cdot u)u^i \tag{4}$$

Direct substitution in these equations yields the following:

Theorem 4. Let (M, g) be a space-time with locally constant connection. The solution of the geodesic equations with initial condition $u^i(0)$ is

$$u^i(t) = \frac{u^i(0) + |u(0)|^2 t\alpha^i}{1 + 2(\alpha \cdot u(0))t + |\alpha|^2|u(0)|^2 t^2} \tag{5}$$

where t is an affine parameter.

If $u^i(0)$ is null (5) reduces to

$$u_N^i(t) = \frac{u^i(0)}{1 + 2(\alpha \cdot u(0))t} \tag{6}$$

The denominator in both (5) and (6) may become zero for finite values of t. In addition $(\alpha \cdot u(t))$ may be zero for finite t. Thus geodesic completeness must be examined.

The denominator of (5) has zeros for

$$t_{\pm} = -(\alpha \cdot u(0)) \pm \left[(\alpha \cdot u(0))^2 - |\alpha|^2 |u(0)|^2\right]^{1/2} \tag{7}$$

while that of (6) has a zero at

$$t_N = -\frac{1}{2(\alpha \cdot u(0))} \tag{8}$$

and $(\alpha \cdot u(t))$ is zero for

$$t_\alpha = -\frac{(\alpha \cdot u(0))}{|\alpha|^2 |u(0)|^2} \tag{9}$$

so long as $u^i(0)$ is not null. At this point we must distinguish the cases in which $u^i(0)$ is null, timelike, or spacelike. In addition, we choose coordinates and normalization of $u^i(t)$ such that $\alpha^i = \delta^i_4$ and $u_i(0)u^i(0) = |u(0)|^2$:

$$u_i u^i = \begin{cases} 1, & u^i \text{ spacelike} \\ 0, & u^i \text{ null} \\ -1, & u^i \text{ timelike} \end{cases} \tag{10}$$

Case I: $u^i(0)$ null. With $\alpha^i = \delta^i_4$ (6) becomes

$$u^i_N(t) = \frac{u^i(0)}{1 - 2u^4(0)t} \tag{11}$$

This is incomplete for

$$t_N = \frac{1}{2u^4(0)} \tag{12}$$

If $u^4(0) > 0$ the geodesic is incomplete to the future. If $u^4(0) < 0$ it is incomplete to the past. If τ is time in the frame $\alpha^i = \delta^i_4$ then

$$u^4_N(t) = \frac{d\tau}{dt} = \frac{u^4(0)}{1 - 2u^4(0)t} \tag{13}$$

giving

$$\tau = -\tfrac{1}{2}\ln\left[1 - 2u^4(0)t\right] \tag{14}$$

Depending on the sign of $u^4(0)$ there will be an event horizon either in the past or in the future. By (13) the frequency of light traveling along a null geodesic undergoes a red shift which approaches infinity as $t \to t_N$.

Case II: $u^i(0)$ spacelike. With $\alpha^i = \delta_4^i$ (5) becomes

$$u^i(t) = \frac{u^i(0) + t\delta_4^i}{1 - 2u^4(0)t - t^2} \tag{15}$$

Observing that for spacelike geodesics $(u^4(0))^2 + 1 = |\mathbf{u}|^2$, where $\mathbf{u}$ is the spacelike part of u^i, (7) requires

$$-|\mathbf{u}| \leqslant t - u^4(0) \leqslant |\mathbf{u}| \tag{16}$$

Note that the domain define by (16) is symmetric if $u^4(0) = 0$, that is, if $(\alpha \cdot u(0)) = 0$. In this case (15) yields

$$\mathbf{u}(t) = \frac{\mathbf{u}(0)}{1 - t^2}$$

$$u^4(t) = \frac{t}{1 - t^2} \tag{17}$$

As t approaches ± 1 spatial distances are contracted to zero and again there is an infinite red shift with respect to the time axis defined by α^i.

Case III: $u^i(0)$ timelike. There are two subcases according to whether $u^i(0)$ is parallel to α^i or not. If $u^i(0)$ is not parallel to α^i (5) becomes

$$u^i(t) = \frac{u^i(0) - t\delta_4^i}{1 - 2u^4(0)t + t^2} \tag{18}$$

and again (7) requires (16). Since $u^i(0)$ is timelike, however, it is not possible that $u^4(t) = 0$. In fact, $u^4(t)$ changes sign at $t = \pm u^4(0)$ [as $u^4(0)$ is, respectively, positive or negative]. If $u^4(0) > 0$, for example, $u^4(t)$ will be positive if $0 < t - u^4(0) < |\mathbf{u}|$ and negative if $-|\mathbf{u}| < t - u^4(0) < 0$. This critical point of the function τ occurs at the t_α of (9). Since τ is defined by $d\tau/dt = u^4(0)$, $t \to u^4(0)$ corresponds to an infinite blue shift. In the past if $u^4(0) < 0$ and in the future if $u^4(0) > 0$. $t \to u^4(0) \pm |\mathbf{u}|$, on the other hand, is an infinite red shift.

If $u^i(0)$ is parallel or antiparallel to α^i [$u^i(0) = \varepsilon\delta_4^i$, $\varepsilon = \pm 1$], then $\mathbf{u} = 0$ and (5) gives us

$$u^i(t) = \frac{\delta_4^i}{\varepsilon - t} \tag{19}$$

If $\varepsilon = 1$ there is an infinite red shift at $t=1$. If $\varepsilon = -1$ this occurs at $t=-1$. In the first case the geodesic is complete to the past. In the second case to the future.

In summary, the geodesics of (5) are incomplete in all directions except the direction either parallel or antiparallel to α and null directions which are future pointing or past pointing, but not both. Extension of geodesics will not be considered in this paper.

Keeping in mind the limits on t we can integrate (5) to obtain, for

$$Q(\alpha, u) = |\alpha|^2 |u(0)|^2 - (\alpha \cdot u(0))^2 \neq 0$$

$$x^i(t) = x^i(0) + \frac{\alpha^i}{2|\alpha|^2} \ln\left[1 + 2(\alpha \cdot u(0))t + |\alpha|^2 |u(0)|^2 t^2\right]$$
$$+ \left[u^i(0) - \frac{(\alpha \cdot u(0))}{|\alpha|^2}\alpha^i\right] H(t) \tag{20}$$

where

$$H(t) = \begin{cases} \dfrac{2}{\sqrt{Q}} \left\{ \tan^{-1}\left[\dfrac{2(|\alpha|^2 |u(0)|^2 t + (\alpha \cdot u(0))}{\sqrt{Q}}\right] - \tan^{-1}\left[\dfrac{2(\alpha \cdot u(0))}{\sqrt{Q}}\right]\right\} & Q > 0 \\[2ex] \dfrac{2}{\sqrt{-Q}} \left\{ \tanh^{-1}\left[\dfrac{2(|\alpha|^2 |u(0)|^2 + (\alpha \cdot u(0))}{\sqrt{-Q}}\right] - \tanh^{-1}\left[\dfrac{2(\alpha \cdot u(0))}{\sqrt{-Q}}\right]\right\} & Q < 0 \end{cases} \tag{21}$$

if $u^i(0)$ is not null, and

$$x^i(t) = x^i(0) + \frac{u^i(0)}{2(\alpha \cdot u(0))} \ln[1 + 2(\alpha \cdot u(0))t] \tag{22}$$

if $u^i(0)$ is null. If $Q(\alpha, u) = 0$ then $u^i(0)$ must be proportional to α^i and

$(\alpha \cdot u(0)) \neq 0$. Equation (5) integrates to give

$$x^i(t) = x^i(0) + \frac{\alpha^i}{|\alpha|^2} \ln\left[\frac{|\alpha|^2 |u(0)|^2 t + (\alpha \cdot u(0))}{(\alpha \cdot u(0))}\right]$$

$$+ \left[u^i(0) - \frac{(\alpha \cdot u(0))}{|\alpha|^2}\alpha^i\right] \frac{|\alpha|^2 u(0)|^2 t}{(\alpha \cdot u(0))\left[|\alpha|^2 |u(0)|^2 t + (\alpha \cdot u(0))\right]} \tag{23}$$

3. GEOMETRIC PROPERTIES

Space-times with locally constant connection exhibit some strikingly elegant geometric properties.

Theorem 5 (Antonelli and Voorhees 1982). Let (M, g) be a conformally flat space-time with locally constant connection. If the vector α^i is not null the Ricci tensor of M has one zero eigenvalue and an eigenvalue $-2e^{-4\phi}(\alpha_r \alpha^r) = -2e^{-2\phi}|\alpha|^2$ of multiplicity three. Further, the eigendirection corresponding to the zero eigenvalue is in the direction α^i.

Proof. The eigenvalue equation for R^i_j is, from (A.3)

$$2e^{-2\phi}\left[(\alpha \cdot k)\alpha^i - |\alpha|^2 k^i\right] = \lambda k^i \tag{24}$$

Clearly $k^i = \alpha^i$ is a solution with $\lambda = 0$. Equally clearly, any k^i satisfying $(\alpha \cdot k) = 0$ is a solution with $\lambda = -2e^{-2\phi}|\alpha|^2$. ■

Since α^i is timelike the three independent eigenvectors corresponding to the $-2e^{-2\phi}|\alpha|^2$ eigenvalue must be spacelike. Let $\{\zeta^i_{(r)} | r = 1, \ldots, 4; \zeta^i_{(4)} = (\alpha_j \cdot \alpha^j)^{-1/2}\alpha^i\}$ be an orthonormal frame of eigenvectors of R_{ij}. The six bivectors

$$w^{ij}_{(rs)} = 2\zeta^{[i}_{(r)}\zeta^{j]}_{(s)} (r, s = 1, \ldots 4; r < s) \tag{25}$$

are a basis for the space of 2-forms on M. It is well known that the Riemann tensor defines an automorphism on this space. Use of (A.2) yields the following:

Theorem 6. The $w_{(rs)}$ of (25) are eigenbivectors of the Riemann tensor. If α^i is not null the eigenvalues are zero and $2e^{-4\phi}(\alpha_r \alpha^r) = 2e^{-2\phi}|\alpha|^2$, each with multiplicity three. The corresponding bivectors are, respectively, $w_{(1s)}$ ($s=2,3,4$) and $w_{(rs)}$ ($r,s=2,3,4$; $r<s$). Each of the $w_{(rs)}$ defines a 2-surface.

Theorem 7. Let $\kappa(r,s)$ be the sectional curvature of the 2-surface defined by $w_{(rs)}$. Then

$$\kappa(1,s)=0 \qquad (s=2,3,4)$$
$$\kappa(r,s)=-2e^{-4\phi}(\alpha_l\alpha^l)=-2e^{-2\phi}|\alpha|^2 \qquad (r,s=2,3,4;\ r<s) \tag{26}$$

Proof follows by direct computation using the formula

$$\kappa(r,s)=-\frac{R_{ijkl}w^{ij}_{(rs)}w^{kl}_{(rs)}}{w^{ij}_{(rs)}w^{(rs)}_{ij}} \tag{27}$$

The 2-surfaces defined by the $w_{(1s)}$ are totally geodesic since by (5) any geodesic which is initially tangent to one of these surfaces remains tangent to that surface. Since the $\zeta^i_{(r)}$ ($r=1,2,3$) can be any set of orthonormal spacelike vectors orthogonal to α^i we obtain the following:

Theorem 8. A conformally flat space-time with locally constant connection admits an $O(2)$ invariant family of totally geodesic 2-surfaces, each with zero sectional curvature.

4. DISCUSSION

In the coordinate system of Section 2 the line element for a conformally flat space of locally constant connection is

$$ds^2=e^{-2\tau}(dx^2+dy^2+dz^2-d\tau^2) \tag{28}$$

where $x=x^1$, $y=x^2$, $z=x^3$, and $\tau=x^4$. The transformation

$$\tau=-\ln\sigma \tag{29}$$

puts this into the form

$$ds^2=\sigma^2(dx^2+dy^2+dz^2)-d\sigma^2 \tag{30}$$

which is the form of a Robinson–Walker universe of infinite radius (Tolman, 1934). Assuming a perfect fluid stress-energy tensor the density and pressure for this universe are given by Tolman as

$$\rho = \frac{3}{\sigma^2}$$

$$p = \frac{-1}{\sigma^2} \tag{31}$$

These expressions can equally well be derived from (A.5). Since the pressure is negative this universe must be considered as unphysical. Note that $\rho + 3p = 0$.

From (29) the time parameter σ goes to infinity as τ goes to *minus* infinity and goes to zero as τ goes to infinity. But $\sigma = 0$ is the big bang for a universe with line element (30). Thus we may think of the space-time described by (28) as a Robinson–Walker universe with negative pressure running in reverse. The infinite blue shift for timelike geodesics discussed in Section 2 then corresponds to the point of maximum contraction.

APPENDIX

The connection coefficients, Riemann, Ricci, and Einstein tensors, and scalar curvature for a space-time with metric (3) are

$$\Gamma^i_{ii} = \alpha^i \qquad (i \neq 4), \qquad \Gamma^4_{44} = -\alpha^4$$

$$\Gamma^i_{ij} = \Gamma^i_{ji} = \alpha^j \qquad (i \neq j,\, j \neq 4), \qquad \Gamma^i_{4i} = -\alpha^4 \qquad (i \neq 4)$$

$$\Gamma^i_{jj} = -\alpha^i \qquad (i \neq j,\, i \neq 4), \qquad \Gamma^4_{jj} = \alpha^4$$

$$\Gamma^i_{jk} = 0, \qquad i \neq j \neq k \tag{A.1}$$

$$R^i_{jkl} = 2\left[\phi,_j\phi,_{[l}\delta^i_{k]} + \eta^{ir}\phi,_r\eta_{j[l}\phi,_{k]} + \eta^{rs}\phi,_r\phi,_s\eta_{j[k}\delta^i_{l]}\right] \tag{A.2}$$

$$R_{ij} = 2(\phi,_i\phi,_j - \eta_{ij}\eta^{rs}\phi,_r\phi,_s) \tag{A.3}$$

$$R = -6e^{-2\phi}\eta^{rs}\phi,_r\phi,_s \tag{A.4}$$

$$G_{ij} = 2\phi,_i\phi,_j + \eta_{ij}\eta^{rs}\phi,_r\phi,_s \tag{A.5}$$

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